

Upon h -normal Γ -linear connections on $J^1(T, M)$

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Abstract

Section 1 introduces the notion of h -normal Γ -linear connection on the 1-jet fibre bundle $J^1(T, M)$, and studies its local components. Section 2 analyses the main local components of torsion and curvature d-tensors attached to an h -normal Γ -linear connection ∇ . Section 3 presents the local Ricci identities induced by ∇ . The identities of the local deflection d-tensors are also exposed. Section 4 is dedicated to the writing of the local Bianchi identities of ∇ .

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1 Components of h -normal Γ -linear connections

Let T (resp. M) be a "*temporal*" (resp. "*spatial*") manifold of dimension p (resp. n) which is coordinated by $(t^\alpha)_{\alpha=\overline{1,p}}$ (resp. $(x^i)_{i=\overline{1,n}}$). Let us consider the 1-jet fibre bundle $J^1(T, M) \rightarrow T \times M$, naturally coordinated by $(t^\alpha, x^i, x_\alpha^i)$. The coordinate transformations on the product manifold $T \times M$, induce the following coordinate transformations (gauge group) on $J^1(T, M)$,

$$(1.1) \quad \begin{cases} \tilde{t}^\alpha = \tilde{t}^\alpha(t^\beta) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}_\alpha^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^\beta}{\partial \tilde{t}^\alpha} x_\beta^j. \end{cases}$$

Note that, throughout this paper, the indices $\alpha, \beta, \gamma, \dots$ run from 1 to p while the indices i, j, k, \dots run from 1 to n .

On $E = J^1(T, M)$, we fixe a nonlinear connection Γ defined by the *temporal* components $M_{(\alpha)\beta}^{(i)}$ and the *spatial* components $N_{(\alpha)j}^{(i)}$. We recall that the transformation rules of the local components of the nonlinear connection Γ are expressed by [10]

$$(1.2) \quad \begin{cases} \tilde{M}_{(\beta)\mu}^{(j)} \frac{\partial \tilde{t}^\mu}{\partial t^\alpha} = M_{(\gamma)\alpha}^{(k)} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}_\beta^j}{\partial t^\alpha} \\ \tilde{N}_{(\beta)k}^{(j)} \frac{\partial \tilde{x}^k}{\partial x^i} = N_{(\gamma)i}^{(k)} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}_\beta^j}{\partial x^i}. \end{cases}$$

Example 1.1 Let $h_{\alpha\beta}(t^\gamma)$ (resp. $g_{ij}(x^k)$) a semi-Riemannian metric on the temporal (resp. spatial) manifold T (resp. M), and $H_{\alpha\beta}^\gamma$ (resp γ_{ij}^k) its Christoffel symbols. Studying the transformation rules of the local components

$$(1.3) \quad \begin{cases} M_{(\beta)\alpha}^{(j)} = -H_{\alpha\beta}^\gamma x_\gamma^j \\ N_{(\beta)i}^{(j)} = \gamma_{ik}^j x_\beta^k, \end{cases}$$

we conclude that $\Gamma_0 = (M_{(\beta)\alpha}^{(j)}, N_{(\beta)i}^{(j)})$ represents a nonlinear connection on E . This is called the *canonical nonlinear connection attached to the semi-Riemannian metrics $h_{\alpha\beta}$ and φ_{ij}* .

Let us consider $\left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right\} \subset \mathcal{X}(E)$ and $\{dt^\alpha, dx^i, \delta x_\alpha^i\} \subset \mathcal{X}^*(E)$ the adapted bases of the nonlinear connection Γ , where

$$(1.4) \quad \begin{cases} \frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} - M_{(\beta)\alpha}^{(j)} \frac{\partial}{\partial x_\beta^j} \\ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(\beta)i}^{(j)} \frac{\partial}{\partial x_\beta^j} \\ \delta x_\alpha^i = dx_\alpha^i + M_{(\alpha)\beta}^{(i)} dt^\beta + N_{(\alpha)j}^{(i)} dx^j. \end{cases}$$

These bases will be used in the description of geometrical objects on E , because their transformation laws are very simple [10]:

$$(1.5) \quad \begin{cases} \frac{\delta}{\delta t^\alpha} = \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \frac{\delta}{\delta \tilde{t}^\beta}, & \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, & \frac{\partial}{\partial x_\alpha^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} \frac{\partial}{\partial \tilde{x}_\beta^j}, \\ dt^\alpha = \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} d\tilde{t}^\beta, & dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j, & \delta x_\alpha^i = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \delta \tilde{x}_\beta^j. \end{cases}$$

In order to develop the theory of Γ -linear connections on the 1-jet space E , we need the following

Proposition 1.1 *i) The Lie algebra $\mathcal{X}(E)$ of vector fields decomposes as*

$$\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}),$$

where

$$\mathcal{X}(\mathcal{H}_T) = \text{Span} \left\{ \frac{\delta}{\delta t^\alpha} \right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(\mathcal{V}) = \text{Span} \left\{ \frac{\partial}{\partial x_\alpha^i} \right\}.$$

ii) The Lie algebra $\mathcal{X}^(E)$ of covector fields decomposes as*

$$\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}),$$

where

$$\mathcal{X}^*(\mathcal{H}_T) = \text{Span}\{dt^\alpha\}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span}\{dx^i\}, \quad \mathcal{X}^*(\mathcal{V}) = \text{Span}\{\delta x_\alpha^i\}.$$

Let us consider h_T , h_M (horizontal) and v (vertical) as the canonical projections of the above decompositions. In this context, we have

Definition 1.1 A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \rightarrow \mathcal{X}(E)$ is called a Γ -linear connection on E if $\nabla h_T = 0$, $\nabla h_M = 0$ and $\nabla v = 0$.

In order to describe in local terms a Γ -linear connection ∇ on E , we need nine unique local components,

$$(1.6) \quad \nabla \Gamma = (\bar{G}_{\beta\gamma}^\alpha, G_{i\gamma}^k, G_{(\alpha)(j)\gamma}^{(i)(\beta)}, \bar{L}_{\beta j}^\alpha, L_{ij}^k, L_{(\alpha)(j)k}^{(i)(\beta)}, \bar{C}_{\beta(k)}^{\alpha(\gamma)}, C_{i(k)}^{j(\gamma)}, C_{(\alpha)(j)(k)}^{(i)(\beta)(\gamma)}),$$

which are locally defined by the relations

$$\begin{aligned} (h_T) \quad \nabla_{\frac{\delta}{\delta t^\gamma}} \frac{\delta}{\delta t^\beta} &= \bar{G}_{\beta\gamma}^\alpha \frac{\delta}{\delta t^\alpha}, \quad \nabla_{\frac{\delta}{\delta t^\gamma}} \frac{\delta}{\delta x^i} = G_{i\gamma}^k \frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\delta}{\delta t^\gamma}} \frac{\partial}{\partial x_\beta^i} = G_{(\alpha)(i)\gamma}^{(k)(\beta)} \frac{\partial}{\partial x_\alpha^k}, \\ (h_M) \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta t^\beta} &= \bar{L}_{\beta j}^\alpha \frac{\delta}{\delta t^\alpha}, \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = L_{ij}^k \frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x_\beta^i} = L_{(\alpha)(i)j}^{(k)(\beta)} \frac{\partial}{\partial x_\alpha^k}, \\ (v) \quad \nabla_{\frac{\partial}{\partial x_\gamma^j}} \frac{\delta}{\delta t^\beta} &= \bar{C}_{\beta(j)}^{\alpha(\gamma)} \frac{\delta}{\delta t^\alpha}, \quad \nabla_{\frac{\partial}{\partial x_\gamma^j}} \frac{\delta}{\delta x^i} = C_{i(j)}^{k(\gamma)} \frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\partial}{\partial x_\gamma^j}} \frac{\partial}{\partial x_\beta^i} = C_{(\alpha)(i)(j)}^{(k)(\beta)(\gamma)} \frac{\partial}{\partial x_\alpha^k}. \end{aligned}$$

Remark 1.1 The transformation rules of the above connection coefficients are completely described in [12].

Example 1.2 Let us consider $\Gamma_0 = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ the canonical nonlinear connection on E , attached to the semi-Riemannian metrics pair $(h_{\alpha\beta}, \varphi_{ij})$. In these conditions, the following local coefficients [12]

$$B\Gamma_0 = (\bar{G}_{\beta\gamma}^\alpha, 0, G_{(\alpha)(i)\gamma}^{(k)(\beta)}, 0, L_{ij}^k, L_{(\alpha)(i)j}^{(k)(\beta)}, 0, 0, 0),$$

where $\bar{G}_{\alpha\beta}^\gamma = H_{\alpha\beta}^\gamma$, $G_{(\gamma)(i)\alpha}^{(k)(\beta)} = -\delta_i^k H_{\alpha\gamma}^\beta$, $L_{ij}^k = \gamma_{ij}^k$ and $L_{(\gamma)(i)j}^{(k)(\beta)} = \delta_\gamma^\beta \gamma_{ij}^k$, verify the transformation rules of the local coefficients of a Γ_0 -linear connection. This is called the *Berwald Γ_0 -linear connection of the metrics pair $(h_{\alpha\beta}, \varphi_{ij})$* .

Now, let ∇ be a Γ -linear connection on E , locally defined by 1.6. The linear connection ∇ induces a natural linear connection on the d-tensors set of the jet fibre bundle $E = J^1(T, M)$, in the following fashion: starting with a vector field X and a d-tensor field D locally expressed by

$$\begin{aligned} X &= X^\alpha \frac{\delta}{\delta t^\alpha} + X^m \frac{\delta}{\delta x^m} + X_{(\alpha)}^{(m)} \frac{\partial}{\partial x_\alpha^m}, \\ D &= D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x_\beta^j} \otimes dt^\gamma \otimes dx^k \otimes \delta x_\delta^l \dots, \end{aligned}$$

we introduce the covariant derivative

$$\begin{aligned} \nabla_X D &= X^\varepsilon \nabla_{\frac{\delta}{\delta t^\varepsilon}} D + X^p \nabla_{\frac{\delta}{\delta x^p}} D + X_{(\varepsilon)}^{(p)} \nabla_{\frac{\partial}{\partial x_\varepsilon^p}} D = \left\{ X^\varepsilon D_{\gamma k(\beta)(l)\dots/\varepsilon}^{\alpha i(j)(\delta)\dots} + X^p \right. \\ &\quad \left. D_{\gamma k(\beta)(l)\dots|_p}^{\alpha i(j)(\delta)\dots} + X_{(\varepsilon)}^{(p)} D_{\gamma k(\beta)(l)\dots|_p}^{\alpha i(j)(\delta)\dots} \right\} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x_\beta^j} \otimes dt^\gamma \otimes dx^k \otimes \delta x_\delta^l \dots, \end{aligned}$$

where

$$\begin{aligned}
(h_T) \quad & \begin{cases} D_{\gamma k(\beta)(l) \dots / \varepsilon}^{\alpha i(j)(\delta) \dots} = \frac{\delta D_{\gamma k(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots}}{\delta t^\varepsilon} + D_{\gamma k(\beta)(l) \dots}^{\mu i(j)(\delta) \dots} \bar{G}_{\mu \varepsilon}^\alpha + \\ D_{\gamma k(\beta)(l) \dots}^{\alpha m(j)(\delta) \dots} G_{m \varepsilon}^i + D_{\gamma k(\mu)(l) \dots}^{\alpha i(m)(\delta) \dots} G_{(\beta)(m) \varepsilon}^{(j)(\mu)} + \dots - \\ - D_{\mu k(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots} \bar{G}_{\gamma \varepsilon}^\mu - D_{\gamma m(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots} G_{k \varepsilon}^m - D_{\gamma k(\beta)(m) \dots}^{\alpha i(j)(\mu) \dots} G_{(\mu)(l) \varepsilon}^{(m)(\delta)} - \dots, \end{cases} \\
(h_M) \quad & \begin{cases} D_{\gamma k(\beta)(l) \dots | p}^{\alpha i(j)(\delta) \dots} = \frac{\delta D_{\gamma k(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots}}{\delta x^p} + D_{\gamma k(\beta)(l) \dots}^{\mu i(j)(\delta) \dots} \bar{L}_{\mu p}^\alpha + \\ D_{\gamma k(\beta)(l) \dots}^{\alpha m(j)(\delta) \dots} L_{m p}^i + D_{\gamma k(\mu)(l) \dots}^{\alpha i(m)(\delta) \dots} L_{(\beta)(m) p}^{(j)(\mu)} + \dots - \\ - D_{\mu k(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots} \bar{L}_{\gamma p}^\mu - D_{\gamma m(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots} L_{k p}^m - D_{\gamma k(\beta)(m) \dots}^{\alpha i(j)(\mu) \dots} L_{(\mu)(l) p}^{(m)(\delta)} - \dots, \end{cases} \\
(v) \quad & \begin{cases} D_{\gamma k(\beta)(l) \dots | (p)}^{\alpha i(j)(\delta) \dots} = \frac{\partial D_{\gamma k(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots}}{\partial x_\varepsilon^p} + D_{\gamma k(\beta)(l) \dots}^{\mu i(j)(\delta) \dots} \bar{C}_{\mu(p)}^{\alpha(\varepsilon)} + \\ D_{\gamma k(\beta)(l) \dots}^{\alpha m(j)(\delta) \dots} C_{m(p)}^{i(\varepsilon)} + D_{\gamma k(\mu)(l) \dots}^{\alpha i(m)(\delta) \dots} C_{(\beta)(m)(p)}^{(j)(\mu)(\varepsilon)} + \dots - \\ - D_{\mu k(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots} \bar{C}_{\gamma(p)}^{\mu(\varepsilon)} - D_{\gamma m(\beta)(l) \dots}^{\alpha i(j)(\delta) \dots} C_{k(p)}^{m(\varepsilon)} - D_{\gamma k(\beta)(m) \dots}^{\alpha i(j)(\mu) \dots} C_{(\mu)(l)(p)}^{(m)(\delta)(\varepsilon)} - \dots. \end{cases}
\end{aligned}$$

The local operators " $/\varepsilon$ ", " $|_p$ " and " $|_{(p)}^{(\varepsilon)}$ " are called the *T-horizontal covariant derivative*, *M-horizontal covariant derivative* and *vertical covariant derivative* of the Γ -linear connection ∇ .

Remarks 1.2 i) In the particular case of a function $f(t^\gamma, x^k, x_\gamma^k)$ on $J^1(T, M)$, the above covariant derivatives reduce to

$$(1.7) \quad \begin{cases} f_{/\varepsilon} = \frac{\delta f}{\delta t^\varepsilon} = \frac{\partial f}{\partial t^\varepsilon} - M_{(\gamma)\varepsilon}^{(k)} \frac{\partial f}{\partial x_\gamma^k} \\ f_{|p} = \frac{\delta f}{\delta x^p} = \frac{\partial f}{\partial x^p} - N_{(\gamma)p}^{(k)} \frac{\partial f}{\partial x_\gamma^k} \\ f|_{(p)}^{(\varepsilon)} = \frac{\partial f}{\partial x_\varepsilon^p}. \end{cases}$$

ii) Particularly, starting with a d-vector field X on $J^1(T, M)$, locally expressed by

$$X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^i \frac{\delta}{\delta x^i} + X_{(\alpha)}^{(i)} \frac{\partial}{\partial x_\alpha^i},$$

the following expressions of above covariant derivatives hold good:

$$(h_T) \quad \begin{cases} X_{/\varepsilon}^\alpha = \frac{\delta X^\alpha}{\delta t^\varepsilon} + X^\mu \bar{G}_{\mu \varepsilon}^\alpha \\ X_{/\varepsilon}^i = \frac{\delta X^i}{\delta t^\varepsilon} + X^m G_{m \varepsilon}^i \\ X_{(\alpha)/\varepsilon}^{(i)} = \frac{\delta X_{(\alpha)}^{(i)}}{\delta t^\varepsilon} + X_{(\mu)}^{(m)} G_{(\alpha)(m) \varepsilon}^{(i)(\mu)}, \end{cases}$$

$$\begin{aligned}
(h_M) \quad & \begin{cases} X|_p^\alpha = \frac{\delta X^\alpha}{\delta x^p} + X^\mu \bar{L}_{\mu p}^\alpha \\ X|_p^i = \frac{\delta X^i}{\delta x^p} + X^m L_{mp}^i \\ X_{(\alpha)|p}^{(i)} = \frac{\delta X_{(\alpha)}^{(i)}}{\delta x^p} + X_{(\mu)}^{(m)} L_{(\alpha)(m)p}^{(i)(\mu)}, \end{cases} \\
(v) \quad & \begin{cases} X^\alpha|_{(p)}^{(\varepsilon)} = \frac{\partial X^\alpha}{\partial x_\varepsilon^p} + X^\mu \bar{C}_{\mu(p)}^{\alpha(\varepsilon)} \\ X^i|_{(p)}^{(\varepsilon)} = \frac{\partial X^i}{\partial x_\varepsilon^p} + X^m C_{m(p)}^{i(\varepsilon)} \\ X_{(\alpha)}^{(i)}|_{(p)}^{(\varepsilon)} = \frac{\partial X_{(\alpha)}^{(i)}}{\partial x_\varepsilon^p} + X_{(\mu)}^{(m)} C_{(\alpha)(m)(p)}^{(i)(\mu)(\varepsilon)}. \end{cases}
\end{aligned}$$

iii) The local covariant derivatives associated to the Berwald Γ_0 -linear connection, will be denoted by " $//_\varepsilon$ ", " \parallel_p " and " $\parallel_{(p)}^{(\varepsilon)}$ ".

Now, let $h_{\alpha\beta}$ be a fixed pseudo-Riemannian metric on the temporal manifold T , $H_{\alpha\beta}^\gamma$ its Christoffel symbols and $J = J_{(\alpha)\beta j}^{(i)} \frac{\partial}{\partial x_\alpha^i} \otimes dt^\beta \otimes dx^j$, where $J_{(\alpha)\beta j}^{(i)} = h_{\alpha\beta} \delta_j^i$, the *normalization d-tensor* [10] attached to the metric $h_{\alpha\beta}$. The big number of coefficients which characterize a Γ -linear connection on E , determines us to consider the following

Definition 1.2 A Γ -linear connection ∇ on $J^1(T, M)$, defined by the local coefficients

$$\nabla\Gamma = (\bar{G}_{\beta\gamma}^\alpha, G_{i\gamma}^k, G_{(\alpha)(i)\gamma}^{(k)(\beta)}, \bar{L}_{\beta j}^\alpha, L_{ij}^k, L_{(\alpha)(i)j}^{(k)(\beta)}, \bar{C}_{\beta(j)}^{\alpha(\gamma)}, C_{i(j)}^{k(\gamma)}, C_{(\alpha)(i)(j)}^{(k)(\beta)(\gamma)}),$$

that verify the relations $\bar{G}_{\beta\gamma}^\alpha = H_{\beta\gamma}^\alpha$, $\bar{L}_{\beta j}^\alpha = 0$, $\bar{C}_{\beta(j)}^{\alpha(\gamma)} = 0$ and $\nabla J = 0$, is called an *h-normal Γ -linear connection*.

Remark 1.3 Taking into account the local covariant T -horizontal " $/_\gamma$ ", M -horizontal " $|_k$ " and vertical " $|_{(k)}^{(\gamma)}$ " covariant derivatives induced by ∇ , the condition $\nabla J = 0$ is equivalent to

$$J_{(\alpha)\beta j / \gamma}^{(i)} = 0, \quad J_{(\alpha)\beta j | k}^{(i)} = 0, \quad J_{(\alpha)\beta j | (k)}^{(i)(\gamma)} = 0.$$

In this context, we can prove the following

Theorem 1.2 *The coefficients of an h-normal Γ -linear connection ∇ verify the identities*

$$\begin{aligned}
\bar{G}_{\alpha\beta}^\gamma &= H_{\alpha\beta}^\gamma, & \bar{L}_{\beta j}^\alpha &= 0, & \bar{C}_{\beta(j)}^{\alpha(\gamma)} &= 0, \\
G_{(\alpha)(i)\gamma}^{(k)(\beta)} &= \delta_\alpha^\beta G_{i\gamma}^k - \delta_i^k H_{\alpha\gamma}^\beta, & L_{(\alpha)(i)j}^{(k)(\beta)} &= \delta_\alpha^\beta L_{ij}^k, & C_{(\alpha)(i)(j)}^{(k)(\beta)(\gamma)} &= \delta_\alpha^\beta C_{i(j)}^{k(\gamma)}.
\end{aligned}$$

Proof. The first three relations come from the definition of an *h-normal Γ -linear connection*.

The condition $\nabla J = 0$ implies locally that

$$\begin{cases} h_{\beta\mu} G_{(\alpha)(j)\gamma}^{(i)(\mu)} = h_{\alpha\beta} G_{j\gamma}^i + \delta_j^i \left[-\frac{\partial h_{\alpha\beta}}{\partial t^\gamma} + H_{\beta\gamma\alpha} \right] \\ h_{\beta\mu} L_{(\alpha)(j)}^{(i)(\mu)} = h_{\alpha\beta} L_{jk}^i \\ h_{\beta\mu} C_{(\alpha)(j)(k)}^{(i)(\mu)(\gamma)} = h_{\alpha\beta} C_{j(k)}^{i(\gamma)}, \end{cases}$$

where $H_{\beta\gamma\alpha} = H_{\beta\gamma}^\mu h_{\mu\alpha}$ represent the Christoffel symbols of the first kind attached to the pseudo-Riemannian metric $h_{\alpha\beta}$. Contracting the above relations by $h^{\beta\varepsilon}$, one obtains the last three identities of the theorem. ■

Remarks 1.4 i) The preceding theorem implies that an h -normal Γ -linear on E is determined just by four effective coefficients

$$(1.8) \quad \nabla\Gamma = (H_{\alpha\beta}^\gamma, G_{i\gamma}^k, L_{ij}^k, C_{i(j)}^{k(\gamma)}).$$

ii) In the particular case $(T, h) = (R, \delta)$, a δ -normal Γ -linear connection identifies to the notion of N -linear connection used in Lagrangian geometry [5].

Example 1.3 The canonical Berwald Γ_0 -linear connection associated to the metrics pair $(h_{\alpha\beta}, \varphi_{ij})$ is an h -normal Γ_0 -linear connection, defined by the local coefficients $B\Gamma_0 = (H_{\alpha\beta}^\gamma, 0, \gamma_{ij}^k, 0)$.

2 Components of torsion and curvature d-tensors

The study of the torsion \mathbf{T} and curvature \mathbf{R} d-tensor of an arbitrary Γ -linear connection ∇ on E was made in [12]. In that context, we proved that the torsion d-tensor is determined by twelve effective local torsion d-tensors, while the curvature d-tensor of ∇ is determined by eighteen local d-tensors.

Let us start with an h -normal Γ -linear connection ∇ . Following the formulas described in [12] and using the properties of ∇ , it follows that the torsion d-tensors $\bar{T}_{\alpha\beta}^\mu$, $\bar{T}_{\alpha j}^\mu$ and $\bar{P}_{\alpha(j)}^{\mu(\beta)}$ vanish. Moreover, we deduce that the following theorem holds good.

Theorem 2.1 *The torsion d-tensor T of an h -normal Γ -linear connection ∇ , is determined by nine effective local d-tensors,*

$$(2.1) \quad \begin{array}{|c|c|c|c|} \hline & h_T & h_M & v \\ \hline h_T h_T & 0 & 0 & R_{(\mu)\alpha\beta}^{(m)} \\ \hline h_M h_T & 0 & T_{\alpha j}^m & R_{(\mu)\alpha j}^{(m)} \\ \hline h_M h_M & 0 & T_{ij}^m & R_{(\mu)ij}^{(m)} \\ \hline v h_T & 0 & 0 & P_{(\mu)\alpha(j)}^{(m)(\beta)} \\ \hline v h_M & 0 & P_{i(j)}^{m(\beta)} & P_{(\mu)i(j)}^{(m)(\beta)} \\ \hline v v & 0 & 0 & S_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)} \\ \hline \end{array}$$

$$\begin{aligned}
\text{where } P_{(\mu)\alpha(j)}^{(m)(\beta)} &= \frac{\partial M_{(\mu)\alpha}^{(m)}}{\partial x_{\beta}^j} - \delta_{\mu}^{\beta} G_{j\alpha}^m + \delta_j^m H_{\mu\alpha}^{\beta}, & P_{(\mu)i(j)}^{(m)(\beta)} &= \frac{\partial N_{(\mu)i}^{(m)}}{\partial x_{\beta}^j} - \delta_{\mu}^{\beta} L_{ji}^m, \\
R_{(\mu)\alpha\beta}^{(m)} &= \frac{\delta M_{(\mu)\alpha}^{(m)}}{\delta t^{\beta}} - \frac{\delta M_{(\mu)\beta}^{(m)}}{\delta t^{\alpha}}, & R_{(\mu)\alpha j}^{(m)} &= \frac{\delta M_{(\mu)\alpha}^{(m)}}{\delta x^j} - \frac{\delta N_{(\mu)j}^{(m)}}{\delta t^{\alpha}}, & R_{(\mu)ij}^{(m)} &= \frac{\delta N_{(\mu)i}^{(m)}}{\delta x^j} - \frac{\delta N_{(\mu)j}^{(m)}}{\delta x^i}, \\
S_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)} &= \delta_{\mu}^{\alpha} C_{i(j)}^{m(\beta)} - \delta_{\mu}^{\beta} C_{j(i)}^{m(\alpha)}, & T_{\alpha j}^m &= -G_{j\alpha}^m, & T_{ij}^m &= L_{ij}^m - L_{ji}^m, & P_{i(j)}^{m(\beta)} &= C_{i(j)}^{m(\beta)}.
\end{aligned}$$

Remark 2.1 For the Berwald Γ_0 -linear connection associated to the metrics $h_{\alpha\beta}$ and φ_{ij} , all torsion d-tensors vanish, except

$$(2.2) \quad R_{(\mu)\alpha\beta}^{(m)} = -H_{\mu\alpha\beta}^{\gamma} x_{\gamma}^m, \quad R_{(\mu)ij}^{(m)} = r_{ijl}^m x_{\mu}^l,$$

where $H_{\mu\alpha\beta}^{\gamma}$ (resp. r_{ijl}^m) are the curvature tensors of the metric $h_{\alpha\beta}$ (resp. φ_{ij}).

The form of expressions of local curvature d-tensors from the general case of a Γ -linear connection [12], and again the properties of the h -normal Γ -linear connection ∇ , imply a reduction (from eighteen to seven) of the number of the effective curvature d-tensors attached to an h -normal Γ -linear connection. Consequently, we obtain

Theorem 2.2 *The curvature d-tensor R of an h -normal Γ -linear connection ∇ , is characterized by seven effective local d-tensors,*

$$(2.3) \quad \begin{array}{|c|c|c|c|} \hline & h_T & h_M & v \\ \hline h_T h_T & H_{\eta\beta\gamma}^{\alpha} & R_{i\beta\gamma}^l & R_{(\eta)(i)\beta\gamma}^{(l)(\alpha)} = \delta_{\eta}^{\alpha} R_{i\beta\gamma}^l + \delta_i^l H_{\eta\beta\gamma}^{\alpha} \\ \hline h_M h_T & 0 & R_{i\beta k}^l & R_{(\eta)(i)\beta k}^{(l)(\alpha)} = \delta_{\eta}^{\alpha} R_{i\beta k}^l \\ \hline h_M h_M & 0 & R_{ijk}^l & R_{(\eta)(i)jk}^{(l)(\alpha)} = \delta_{\eta}^{\alpha} R_{ijk}^l \\ \hline v h_T & 0 & P_{i\beta(k)}^{l(\gamma)} & P_{(\eta)(i)\beta(k)}^{(l)(\alpha)(\gamma)} = \delta_{\eta}^{\alpha} P_{i\beta(k)}^{l(\gamma)} \\ \hline v h_M & 0 & P_{ij(k)}^{l(\gamma)} & P_{(\eta)(i)j(k)}^{(l)(\alpha)(\gamma)} = \delta_{\eta}^{\alpha} P_{ij(k)}^{l(\gamma)} \\ \hline v v & 0 & S_{i(j)(k)}^{l(\beta)(\gamma)} & S_{(\eta)(i)(j)(k)}^{(l)(\alpha)(\beta)(\gamma)} = \delta_{\eta}^{\alpha} S_{i(j)(k)}^{l(\beta)(\gamma)} \\ \hline \end{array}$$

where

$$\begin{aligned}
H_{\eta\beta\gamma}^{\alpha} &= \frac{\partial H_{\eta\beta}^{\alpha}}{\partial t^{\gamma}} - \frac{\partial H_{\eta\gamma}^{\alpha}}{\partial t^{\beta}} + H_{\eta\beta}^{\mu} H_{\mu\gamma}^{\alpha} - H_{\eta\gamma}^{\mu} H_{\mu\beta}^{\alpha}, \\
R_{i\beta\gamma}^l &= \frac{\delta G_{i\beta}^l}{\delta t^{\gamma}} - \frac{\delta G_{i\gamma}^l}{\delta t^{\beta}} + G_{i\beta}^m G_{m\gamma}^l - G_{i\gamma}^m G_{m\beta}^l + C_{i(m)}^{l(\mu)} R_{(\mu)\beta\gamma}^{(m)}, \\
R_{i\beta k}^l &= \frac{\delta G_{i\beta}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta t^{\beta}} + G_{i\beta}^m L_{mk}^l - L_{ik}^m G_{m\beta}^l + C_{i(m)}^{l(\mu)} R_{(\mu)\beta k}^{(m)}, \\
R_{ijk}^l &= \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^m L_{mk}^l - L_{ik}^m L_{mj}^l + C_{i(m)}^{l(\mu)} R_{(\mu)jk}^{(m)}, \\
P_{i\beta(k)}^{l(\gamma)} &= \frac{\partial G_{i\beta}^l}{\partial x_{\gamma}^k} - C_{i(k)/\beta}^{l(\gamma)} + C_{i(m)}^{l(\mu)} P_{(\mu)\beta(k)}^{(m)(\gamma)}, \\
P_{ij(k)}^{l(\gamma)} &= \frac{\partial L_{ij}^l}{\partial x_{\gamma}^k} - C_{i(k)|j}^{l(\gamma)} + C_{i(m)}^{l(\mu)} P_{(\mu)j(k)}^{(m)(\gamma)},
\end{aligned}$$

$$S_{i(j)(k)}^{l(\beta)(\gamma)} = \frac{\partial C_{i(j)}^{l(\beta)}}{\partial x_\gamma^k} - \frac{\partial C_{i(k)}^{l(\gamma)}}{\partial x_\beta^j} + C_{i(j)}^{m(\beta)} C_{m(k)}^{l(\gamma)} - C_{i(k)}^{m(\gamma)} C_{m(j)}^{l(\beta)}.$$

Remark 2.1 In the case of the Berwald Γ_0 -linear connection associated to the metric pair $(h_{\alpha\beta}, \varphi_{ij})$, all curvature d-tensors vanish, except $H_{\alpha\beta\gamma}^\delta$ and $R_{ijk}^l = r_{ijk}^l$, where r_{ijk}^l are the curvature tensors of the metric φ_{ij} .

3 Ricci identities. Deflection d-tensors identities

The Ricci identities of a Γ -linear connection are described in [12]. In the particular case of an h -normal Γ -linear connection, these simplify because the number and the form of the torsion and curvature d-tensors reduced. A meaningful reduction of these identities can be obtained, considering the more particular case of an h -normal Γ -linear connection ∇ of *Cartan type*, (i. e. , $L_{jk}^i = L_{kj}^i$ and $C_{j(k)}^{i(\gamma)} = C_{k(j)}^{i(\gamma)}$). In that case, the condition $L_{jk}^i = L_{kj}^i$ implies $T_{jk}^i = 0$. Consequently, we have

Theorem 3.1 *The following Ricci identities of an h -normal Γ -linear connection of Cartan type, are true:*

$$\begin{aligned} (h_T) \quad & \left\{ \begin{array}{l} X_{/\beta/\gamma}^\alpha - X_{/\gamma/\beta}^\alpha = X^\mu H_{\mu\beta\gamma}^\alpha - X^\alpha|_{(m)}^{(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\ X_{/\beta|k}^\alpha - X_{|k/\beta}^\alpha = -X_{|m}^\alpha T_{\beta k}^m - X^\alpha|_{(m)}^{(\mu)} R_{(\mu)\beta k}^{(m)} \\ X_{|j|k}^\alpha - X_{|k|j}^\alpha = -X^\alpha|_{(m)}^{(\mu)} R_{(\mu)jk}^{(m)} \\ X_{/\beta|k}^\alpha - X_{|k/\beta}^\alpha = -X^\alpha|_{(m)}^{(\mu)} P_{(\mu)\beta(k)}^{(m)} \\ X_{|j|k}^\alpha - X_{|k|j}^\alpha = -X_{|m}^\alpha C_{j(k)}^{m(\gamma)} - X^\alpha|_{(m)}^{(\mu)} P_{(\mu)j(k)}^{(m)} \\ X^\alpha|_{(j)}^{(\beta)}|_{(k)}^{(\gamma)} - X^\alpha|_{(k)}^{(\gamma)}|_{(j)}^{(\beta)} = -X^\alpha|_{(m)}^{(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)}, \end{array} \right. \\ (h_M) \quad & \left\{ \begin{array}{l} X_{/\beta/\gamma}^i - X_{/\gamma/\beta}^i = X^m R_{m\beta\gamma}^i - X^i|_{(m)}^{(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\ X_{/\beta|k}^i - X_{|k/\beta}^i = X^m R_{m\beta k}^i - X_{|m}^i T_{\beta k}^m - X^i|_{(m)}^{(\mu)} R_{(\mu)\beta k}^{(m)} \\ X_{|j|k}^i - X_{|k|j}^i = X^m R_{mj k}^i - X^i|_{(m)}^{(\mu)} R_{(\mu)jk}^{(m)} \\ X_{/\beta|k}^i - X_{|k/\beta}^i = X^m P_{m\beta(k)}^i - X^i|_{(m)}^{(\mu)} P_{(\mu)\beta(k)}^{(m)} \\ X_{|j|k}^i - X_{|k|j}^i = X^m P_{mj(k)}^i - X_{|m}^i C_{j(k)}^{m(\gamma)} - X^i|_{(m)}^{(\mu)} P_{(\mu)j(k)}^{(m)} \\ X^\alpha|_{(j)}^{(\beta)}|_{(k)}^{(\gamma)} - X^\alpha|_{(k)}^{(\gamma)}|_{(j)}^{(\beta)} = X^m S_{m(j)(k)}^{i(\beta)(\gamma)} - X^i|_{(m)}^{(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)}, \end{array} \right. \end{aligned}$$

$$(v) \quad \left\{ \begin{array}{l} X_{(\alpha)/\beta/\gamma}^{(i)} - X_{(\alpha)/\gamma/\beta}^{(i)} = X_{(\alpha)}^{(m)} R_{m\beta\gamma}^i - X_{(\mu)}^{(i)} H_{\alpha\beta\gamma}^\mu - X_{(\alpha)}^{(i)}|_{(m)}^{(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\ X_{(\alpha)/\beta|k}^{(i)} - X_{(\alpha)|k/\beta}^{(i)} = X_{(\alpha)}^{(m)} R_{m\beta k}^i - X_{(\alpha)|m}^{(i)} T_{\beta k}^m - X_{(\alpha)}^{(i)}|_{(m)}^{(\mu)} R_{(\mu)\beta k}^{(m)} \\ X_{(\alpha)|j|k}^{(i)} - X_{(\alpha)|k|j}^{(i)} = X_{(\alpha)}^{(m)} R_{mjk}^i - X_{(\alpha)}^{(i)}|_{(m)}^{(\mu)} R_{(\mu)jk}^{(m)} \\ X_{(\alpha)/\beta|k}^{(i)}|_{(k)}^{(\gamma)} - X_{(\alpha)|k/\beta}^{(i)}|_{(k)}^{(\gamma)} = X_{(\alpha)}^{(m)} P_{m\beta(k)}^i{}^{(\gamma)} - X_{(\alpha)}^{(i)}|_{(m)}^{(\mu)} P_{(\mu)\beta(k)}^{(m)}{}^{(\gamma)} \\ X_{(\alpha)|j|k}^{(i)}|_{(k)}^{(\gamma)} - X_{(\alpha)|k|j}^{(i)}|_{(k)}^{(\gamma)} = X_{(\alpha)}^{(m)} P_{mj(k)}^i{}^{(\gamma)} - X_{(\alpha)|m}^{(i)} C_{j(k)}^{m(\gamma)} - X_{(\alpha)}^{(i)}|_{(m)}^{(\mu)} P_{(\mu)j(k)}^{(m)}{}^{(\gamma)} \\ X_{(\alpha)}^{(i)}|_{(j)}^{(\beta)}|_{(k)}^{(\gamma)} - X_{(\alpha)}^{(i)}|_{(k)}^{(\gamma)}|_{(j)}^{(\beta)} = X_{(\alpha)}^{(m)} S_{m(j)(k)}^{i(\beta)(\gamma)} - X_{(\alpha)}^{(i)}|_{(m)}^{(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)}, \end{array} \right.$$

where $X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^i \frac{\delta}{\delta x^i} + X_{(\alpha)}^{(i)} \frac{\partial}{\partial x_\alpha^i}$ is an arbitrary d -vector field on $J^1(T, M)$.

In the sequel, let us consider the canonical Liouville d -tensor $\mathbf{C} = x_\alpha^i \frac{\partial}{\partial x_\alpha^i}$. We construct the *deflection d -tensors associated to the h -normal Γ -linear connection ∇* , setting

$$(3.1) \quad \bar{D}_{(\alpha)\beta}^{(i)} = x_\alpha^i{}_{/\beta}, \quad D_{(\alpha)k}^{(i)} = x_\alpha^i{}_{|k}, \quad d_{(\alpha)(k)}^{(i)(\gamma)} = x_\alpha^i{}_{|k}^{(\gamma)},$$

where " $_{/\beta}$ ", " $_{|k}$ " and " $_{|k}^{(\gamma)}$ " are the local covariant derivatives induced by ∇ . By a direct calculation, the deflection d -tensors get the expressions

$$(3.2) \quad \left\{ \begin{array}{l} \bar{D}_{(\alpha)\beta}^{(i)} = -M_{(\alpha)\beta}^{(i)} + G_{m\beta}^i x_\alpha^m - H_{\alpha\beta}^\mu x_\mu^i \\ D_{(\alpha)j}^{(i)} = -N_{(\alpha)j}^{(i)} + L_{mj}^i x_\alpha^m \\ d_{(\alpha)(j)}^{(i)(\beta)} = \delta_j^\beta \delta_\alpha^i + C_{m(j)}^{i(\beta)} x_\alpha^m. \end{array} \right.$$

Applying the (v) -set of the Ricci identities to the components of the Liouville vector field, we obtain

Theorem 3.2 *The deflection d -tensors, attached to the h -normal Γ -linear connection ∇ , verify the identities:*

$$(3.3) \quad \left\{ \begin{array}{l} \bar{D}_{(\alpha)\beta/\gamma}^{(i)} - \bar{D}_{(\alpha)\gamma/\beta}^{(i)} = x_\alpha^m R_{m\beta\gamma}^i - x_\mu^i H_{\alpha\beta\gamma}^\mu - d_{(\alpha)(m)}^{(i)(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\ \bar{D}_{(\alpha)\beta|k}^{(i)} - D_{(\alpha)k/\beta}^{(i)} = x_\alpha^m R_{m\beta k}^i - D_{(\alpha)m}^{(i)} T_{\beta k}^m - d_{(\alpha)(m)}^{(i)(\mu)} R_{(\mu)\beta k}^{(m)} \\ D_{(\alpha)j|k}^{(i)} - D_{(\alpha)k|j}^{(i)} = x_\alpha^m R_{mjk}^i - d_{(\alpha)(m)}^{(i)(\mu)} R_{(\mu)jk}^{(m)} \\ \bar{D}_{(\alpha)\beta|k}^{(i)}|_{(k)}^{(\gamma)} - d_{(\alpha)(k)/\beta}^{(i)(\gamma)} = x_\alpha^m P_{m\beta(k)}^i{}^{(\gamma)} - d_{(\alpha)(m)}^{(i)(\mu)} P_{(\mu)\beta(k)}^{(m)}{}^{(\gamma)} \\ D_{(\alpha)j|k}^{(i)}|_{(k)}^{(\gamma)} - d_{(\alpha)(k)|j}^{(i)(\gamma)} = x_\alpha^m P_{mj(k)}^i{}^{(\gamma)} - D_{(\alpha)m}^{(i)} C_{j(k)}^{m(\gamma)} - d_{(\alpha)(m)}^{(i)(\mu)} P_{(\mu)j(k)}^{(m)}{}^{(\gamma)} \\ d_{(\alpha)(j)|k}^{(i)(\beta)}|_{(k)}^{(\gamma)} - d_{(\alpha)(k)|j}^{(i)(\gamma)}|_{(j)}^{(\beta)} = x_\alpha^m S_{m(j)(k)}^{i(\beta)(\gamma)} - d_{(\alpha)(m)}^{(i)(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)}. \end{array} \right.$$

Remark 3.1 The importance of the deflection d -tensors identities is emphasized in [7], [9], where is developed the (*generalized*) *metrical multi-time Lagrangian geometry*

of physical fields on $J^1(T, M)$. In that context, the deflection d-tensors identities are used in the description of the Maxwell equations which govern the electromagnetic field of a (*generalized*) *metrical multi-time Lagrange space* [7], [11].

4 Bianchi identities

From the general theory of linear connections on a vector bundle E , is known that the torsions \mathbf{T} and the curvature \mathbf{R} of a linear connection ∇ are not independent. They verify the Bianchi identities, whose expressions, in a local basis (X_A) of $\mathcal{X}(E)$, are [5], [12]

$$(4.1) \quad \begin{cases} \sum_{\{A,B,C\}} \{R_{ABC}^F - T_{AB:C}^F - T_{AB}^G T_{CG}^F\} = 0 \\ \sum_{\{A,B,C\}} \{R_{DAB:C}^F + T_{AB}^G R_{DAG}^F\} = 0, \end{cases}$$

where $\mathbf{R}(X_A, X_B)X_C = R_{CBA}^D X_D$, $\mathbf{T}(X_A, X_B) = T_{BA}^D X_D$ and " :C " represents the local covariant derivative induced by ∇ .

In our context, we have $E = J^1(T, M)$. Let $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ be a fixed nonlinear connection on E , and $(X_A) = \left(\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right)$ its adapted basis. In the sequel, we try to rewrite the above Bianchi identities for an h -normal Γ -linear connection ∇ of Cartan type on E . In this sense, taking into account that the indices A, B, \dots are of type $\left\{ \alpha, i, \binom{(\alpha)}{(i)} \right\}$, it follows that the covariant derivative " :A " becomes one of the coveriant derivatives " / $_\alpha$ ", " | $_i$ " or " | $_{(i)}^{(\alpha)}$ ". Consequently, we deduce

Theorem 4.1 *The following thirty effective Bianchi identities of the h -normal Γ -linear connection ∇ of Cartan type are true:*

$$(1) \quad \left\{ \begin{array}{l} 1.1 \quad \sum_{\{\alpha,\beta,\gamma\}} H_{\alpha\beta\gamma}^\delta = 0 \\ 1.2 \quad \mathcal{A}_{\{\alpha,\beta\}} \left\{ T_{\alpha m}^l T_{\beta k}^m - T_{\alpha k/\beta}^l \right\} = R_{k\alpha\beta}^l - C_{k(m)}^{l(\mu)} R_{(\mu)\alpha\beta}^{(m)} \\ 1.3 \quad \mathcal{A}_{\{j,k\}} \left\{ C_{k(m)}^{l(\mu)} R_{(\mu)\alpha j}^{(m)} + R_{j\alpha k}^l + T_{\alpha j|k}^l \right\} = 0 \\ 1.4 \quad \sum_{\{i,j,k\}} \left\{ C_{k(m)}^{l(\mu)} R_{(\mu)ij}^{(m)} - R_{ijk}^l \right\} = 0, \end{array} \right.$$

$$(2) \quad \left\{ \begin{array}{l} 2.1 \quad \sum_{\{\alpha,\beta,\gamma\}} \left\{ R_{(\delta)\alpha\beta/\gamma}^{(l)} + P_{(\delta)\gamma(m)}^{(l)(\mu)} R_{(\mu)\alpha\beta}^{(m)} \right\} = 0 \\ 2.2 \quad \mathcal{A}_{\{\alpha,\beta\}} \left\{ R_{(\delta)\alpha k/\beta}^{(l)} + P_{(\delta)\beta(m)}^{(l)(\mu)} R_{(\mu)\alpha k}^{(m)} + R_{(\delta)\beta m}^{(l)} T_{\alpha k}^m \right\} = R_{(\delta)\alpha\beta|k}^{(l)} + \\ \quad \quad \quad + P_{(\delta)k(m)}^{(l)(\mu)} R_{(\mu)\alpha\beta}^{(m)} \\ 2.3 \quad \mathcal{A}_{\{j,k\}} \left\{ R_{(\delta)\alpha j|k}^{(l)} + P_{(\delta)k(m)}^{(l)(\mu)} R_{(\mu)\alpha j}^{(m)} + R_{(\delta)km}^{(l)} T_{\alpha j}^m \right\} = -R_{(\delta)\alpha j|k}^{(l)} - \\ \quad \quad \quad - P_{(\delta)\alpha(m)}^{(l)(\mu)} R_{(\mu)jk}^{(m)} \\ 2.4 \quad \sum_{\{i,j,k\}} \left\{ R_{(\delta)ij|k}^{(l)} + P_{(\delta)k(m)}^{(l)(\mu)} R_{(\mu)ij}^{(m)} \right\} = 0, \end{array} \right.$$

$$(3) \quad \left\{ \begin{array}{l} 3.1 \quad T_{\alpha k}^l |_{(p)}^{(\varepsilon)} - C_{m(p)}^{l(\varepsilon)} T_{\alpha k}^m + P_{k\alpha(p)}^l |_{(\varepsilon)} - C_{k(p)/\alpha}^{l(\varepsilon)} - C_{k(m)}^{l(\mu)} P_{(\mu)\alpha(p)}^{(m) (\varepsilon)} = 0 \\ 3.2 \quad \mathcal{A}_{\{j,k\}} \left\{ C_{j(p)|k}^{l(\varepsilon)} + C_{k(m)}^{l(\mu)} P_{(\mu)j(p)}^{(m) (\varepsilon)} + P_{jk(p)}^l |_{(\varepsilon)} \right\} = 0, \end{array} \right.$$

$$(4) \quad \left\{ \begin{array}{l} 4.1 \quad \mathcal{A}_{\{\alpha,\beta\}} \left\{ P_{(\delta)\alpha(p)/\beta}^{(l) (\varepsilon)} + P_{(\delta)\beta(m)}^{(l) (\mu)} P_{(\mu)\alpha(p)}^{(m) (\varepsilon)} \right\} = R_{(\delta)\alpha\beta}^{(l)} |_{(p)}^{(\varepsilon)} - \\ \quad - R_{(\delta)(p)\alpha\beta}^{(l)(\varepsilon)} + S_{(\delta)(p)(m)}^{(l)(\varepsilon)(\mu)} R_{(\mu)\alpha\beta}^{(m)} \\ 4.2 \quad \mathcal{A}_{\{\alpha,k\}} \left\{ P_{(\delta)\alpha(p)|k}^{(l) (\varepsilon)} + P_{(\delta)k(m)}^{(l) (\mu)} P_{(\mu)\alpha(p)}^{(m) (\varepsilon)} \right\} = R_{(\delta)\alpha k}^{(l)} |_{(p)}^{(\varepsilon)} - \\ \quad - R_{(\delta)(p)\alpha k}^{(l)(\varepsilon)} + S_{(\delta)(p)(m)}^{(l)(\varepsilon)(\mu)} R_{(\mu)\alpha k}^{(m)} + R_{(\delta)\alpha m}^{(l)} C_{k(p)}^{m(\varepsilon)} - T_{\alpha k}^m P_{(\delta)m(p)}^{(l) (\varepsilon)} \\ 4.3 \quad \mathcal{A}_{\{j,k\}} \left\{ P_{(\delta)j(p)|k}^{(l) (\varepsilon)} + P_{(\delta)k(m)}^{(l) (\mu)} P_{(\mu)j(p)}^{(m) (\varepsilon)} + R_{(\delta)km}^{(l)} C_{j(p)}^{m(\varepsilon)} \right\} = \\ \quad = R_{(\delta)jk}^{(l)} |_{(p)}^{(\varepsilon)} - R_{(\delta)(p)jk}^{(l)(\varepsilon)} + S_{(\delta)(p)(m)}^{(l)(\varepsilon)(\mu)} R_{(\mu)jk}^{(m)}, \end{array} \right.$$

$$(5) \quad \left\{ \begin{array}{l} 5.1 \quad \mathcal{A}_{\left\{ \begin{smallmatrix} (\beta) \\ (j) \end{smallmatrix}, \begin{smallmatrix} (\gamma) \\ (k) \end{smallmatrix} \right\}} \left\{ C_{i(j)}^{l(\beta)} |_{(k)}^{(\gamma)} + C_{i(k)}^{m(\gamma)} C_{m(j)}^{l(\beta)} \right\} = S_{i(j)(k)}^{l(\beta)(\gamma)} - C_{i(m)}^{l(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)}, \end{array} \right.$$

$$(6) \quad \left\{ \begin{array}{l} 6.1 \quad \mathcal{A}_{\left\{ \begin{smallmatrix} (\beta) \\ (j) \end{smallmatrix}, \begin{smallmatrix} (\gamma) \\ (k) \end{smallmatrix} \right\}} \left\{ P_{(\delta)\alpha(j)}^{(l) (\beta)} |_{(k)}^{(\gamma)} + P_{(\mu)\alpha(j)}^{(m) (\beta)} S_{(\delta)(k)(m)}^{(l)(\gamma)(\mu)} + P_{(\delta)(j)\alpha(k)}^{(l)(\beta) (\gamma)} \right\} = \\ \quad = -S_{(\delta)(j)(k)/\alpha}^{(l)(\beta)(\gamma)} - S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)} P_{(\delta)\alpha(m)}^{(l) (\mu)} \\ 6.2 \quad \mathcal{A}_{\left\{ \begin{smallmatrix} (\beta) \\ (j) \end{smallmatrix}, \begin{smallmatrix} (\gamma) \\ (k) \end{smallmatrix} \right\}} \left\{ P_{(\delta)i(j)}^{(l) (\beta)} |_{(k)}^{(\gamma)} + P_{(\mu)i(j)}^{(m) (\beta)} S_{(\delta)(k)(m)}^{(l)(\gamma)(\mu)} + P_{(\delta)(j)i(k)}^{(l)(\beta) (\gamma)} \right\} = \\ \quad = -S_{(\delta)(j)(k)|i}^{(l)(\beta)(\gamma)} - S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)} P_{(\delta)i(m)}^{(l) (\mu)}, \end{array} \right.$$

$$(7) \quad \left\{ \begin{array}{l} 7.1 \quad \sum_{\left\{ \begin{smallmatrix} (\alpha) \\ (i) \end{smallmatrix}, \begin{smallmatrix} (\beta) \\ (j) \end{smallmatrix}, \begin{smallmatrix} (\gamma) \\ (k) \end{smallmatrix} \right\}} \left\{ S_{(\delta)i(j)}^{(l)(\alpha)(\beta)} |_{(k)}^{(\gamma)} + S_{(\mu)i(j)}^{(m)(\alpha)(\beta)} S_{(\delta)(k)(m)}^{(l)(\gamma)(\mu)} - S_{(\delta)i(j)(k)}^{(l)(\alpha)(\beta)(\gamma)} \right\} = 0, \end{array} \right.$$

$$(8) \quad \left\{ \begin{array}{l} 8.1 \quad \sum_{\{\alpha,\beta,\gamma\}} H_{\varepsilon\alpha\beta/\gamma}^\delta = 0 \\ 8.2 \quad H_{\varepsilon\alpha\beta|k}^\delta = 0 \\ 8.3 \quad \sum_{\{i,j,k\}} R_{(\mu)ij}^{(m)} P_{(\varepsilon)k(m)}^{(\delta) (\mu)} = 0 \\ 8.4 \quad \sum_{\{\alpha,\beta,\gamma\}} \left\{ R_{p\alpha\beta/\gamma}^l - R_{(\mu)\alpha\beta}^{(m)} P_{p\gamma(m)}^l |_{(\mu)} \right\} = 0 \\ 8.5 \quad \mathcal{A}_{\{\alpha,\beta\}} \left\{ R_{p\alpha k/\beta}^l + R_{(\mu)\alpha k}^{(m)} P_{p\beta(m)}^l |_{(\mu)} - T_{\alpha k}^m R_{p\beta m}^l \right\} = R_{p\alpha\beta}^l + R_{(\mu)\alpha\beta}^{(m)} P_{pk(m)}^l |_{(\mu)} \\ 8.6 \quad \mathcal{A}_{\{j,k\}} \left\{ R_{p\alpha j|k}^l + R_{(\mu)\alpha j}^{(m)} P_{pk(m)}^l |_{(\mu)} - T_{\alpha j}^m R_{pk m}^l \right\} = -R_{pj k/\alpha}^l + R_{(\mu)\alpha k}^{(m)} P_{pj(m)}^l |_{(\mu)} \\ 8.7 \quad \sum_{\{i,j,k\}} \left\{ R_{pij|k}^l - R_{(\mu)ij}^{(m)} P_{pk(m)}^l |_{(\mu)} \right\} = 0, \end{array} \right.$$

$$\begin{aligned}
(9) \quad & \left\{ \begin{array}{l} 9.1 \quad \mathcal{A}_{\{\alpha, \beta\}} \left\{ P_{i\alpha(p)/\beta}^l(\varepsilon) - P_{(\mu)\alpha(p)}^{(m)(\varepsilon)} P_{i\beta(m)}^l(\mu) \right\} = R_{i\alpha\beta}^l|_{(p)}^{(\varepsilon)} + R_{(\mu)\alpha\beta}^{(m)} S_{i(p)(m)}^{l(\varepsilon)(\mu)} \\ 9.2 \quad \mathcal{A}_{\{\alpha, k\}} \left\{ P_{i\alpha(p)|k}^l(\varepsilon) - P_{(\mu)\alpha(p)}^{(m)(\varepsilon)} P_{ik(m)}^l(\mu) \right\} = R_{i\alpha k}^l|_{(p)}^{(\varepsilon)} - R_{(\mu)\alpha k}^{(m)} S_{i(p)(m)}^{l(\varepsilon)(\mu)} - \\ \quad - C_{k(p)}^{m(\varepsilon)} R_{i\alpha m}^l + T_{\alpha k}^m P_{im(p)}^l(\varepsilon) \\ 9.3 \quad \mathcal{A}_{\{j, k\}} \left\{ P_{ij(p)|k}^l(\varepsilon) - P_{(\mu)j(p)}^{(m)(\varepsilon)} P_{ik(m)}^l(\mu) - C_{j(p)}^{m(\varepsilon)} R_{ikm}^l \right\} = R_{ijk}^l|_{(p)}^{(\varepsilon)} + \\ \quad + R_{(\mu)jk}^{(m)} S_{i(p)(m)}^{l(\varepsilon)(\mu)}, \end{array} \right. \\
(10) \quad & \left\{ \begin{array}{l} 10.1 \quad \mathcal{A}_{\left\{ \begin{smallmatrix} (\beta) \\ (j) \end{smallmatrix}, \begin{smallmatrix} (\gamma) \\ (k) \end{smallmatrix} \right\}} \left\{ P_{p\alpha(j)|k}^{l(\beta)(\gamma)} - P_{(\mu)\alpha(j)}^{(m)(\beta)} S_{p(k)(m)}^{l(\gamma)(\mu)} \right\} = S_{p(j)(k)/\alpha}^{l(\beta)(\gamma)} + \\ \quad + S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)} P_{p\alpha(m)}^l(\mu) \\ 10.2 \quad \mathcal{A}_{\left\{ \begin{smallmatrix} (\beta) \\ (j) \end{smallmatrix}, \begin{smallmatrix} (\gamma) \\ (k) \end{smallmatrix} \right\}} \left\{ P_{pi(j)|k}^{l(\beta)(\gamma)} - P_{(\mu)i(j)}^{(m)(\beta)} S_{p(k)(m)}^{l(\gamma)(\mu)} + C_{i(j)}^{m(\beta)} P_{pm(k)}^l(\gamma) \right\} = \\ \quad = S_{p(j)(k)|i}^{l(\beta)(\gamma)} + S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)} P_{pi(m)}^l(\mu), \end{array} \right. \\
(11) \quad & \left\{ \begin{array}{l} 11.1 \quad \sum_{\left\{ \begin{smallmatrix} (\alpha) \\ (i) \end{smallmatrix}, \begin{smallmatrix} (\beta) \\ (j) \end{smallmatrix}, \begin{smallmatrix} (\gamma) \\ (k) \end{smallmatrix} \right\}} \left\{ S_{p(i)(j)|k}^{l(\alpha)(\beta)(\gamma)} + S_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)} S_{p(k)(m)}^{l(\gamma)(\mu)} \right\} = 0, \end{array} \right.
\end{aligned}$$

where, if $\{A, B, C\}$ are indices of type $\left\{ \alpha, i, \begin{smallmatrix} (\alpha) \\ (i) \end{smallmatrix} \right\}$ then $\sum_{\{A, B, C\}}$ means the cyclic sum and $\mathcal{A}_{\{A, B\}}$ means alternate sum.

Remarks 4.1 i) In the particular case $(T, h) = (R, \delta)$, the last identity of every above set of the Bianchi identities reduces to the one of the classical eleven Bianchi identities of an N -linear connection from the Lagrange geometry [5].

ii) The Bianchi identities of an h -normal Γ -linear connection of Cartan type are used in the description of the Maxwell equations and the conservation laws of the Einstein equations of the gravitational potentials from the background of the (generalized) metrical multi-time Lagrange geometry of physical fields [7], [9].

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References

- [1] G. S. Asanov, *Gauge-Covariant Stationary Curves on Finslerian and Jet Fibrations and Gauge Extension of Lorentz Force*, Tensor N. S. **(50)** (1991), 122-137.
- [2] L. A. Cordero, C. T. J. Dodson, M. de Léon, *Differential Geometry of Frame Bundles*, Kluwer Academic Publishers, 1989.
- [3] M. J. Gotay, J. Isenberg, J. E. Marsden, *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Fields*, <http://xxx.lanl.gov/hep/9801019>, 1998.

- [4] N. Kamron et P. J. Olver, *Le Problème d'équivalence à une divergence près dans le calcul des variations des intégrales multiples*, C. R. Acad. Sci. Paris, t. 308, Série I, p. 249-252, 1989.
- [5] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.
- [6] R. Miron, M. S. Kirkovits, M. Anastasiei, *A Geometrical Model for Variational Problems of Multiple Integrals*, Proc. of Conf. of Diff. Geom. and Appl. , June 26-July 3, 1988, Dubrovnik, Yugoslavia.
- [7] M. Neagu, *Generalized Metrical Multi-Time Lagrange Geometry of Physical Fields*, 2000, to appear.
- [8] M. Neagu, *Harmonic Maps between Generalized Lagrange Spaces*, Southeast Asian Bulletin of Mathematics, Springer-Verlag, 2000-2001, in press.
- [9] M. Neagu, *Metrical Multi-Time Lagrangian Geometry of Physical Fields*, Workshop on Diff. Geom. , Global Analysis, Lie Algebras, Aristotle University of Thessaloniki, Greece, Aug. 27-Sept. 2, 2000; <http://xxx.lanl.gov/math.DG/0009117>, 2000.
- [10] M. Neagu, C. Udriște, *Geometrical Objects on Jet Fibre Bundle of Order One*, Third Conference of Balkan Society of Geometers, Politehnica University of Bucharest, Romania, July 31-August 3, 2000; <http://xxx.lanl.gov/math.DG/0009049>, 2000.
- [11] M. Neagu, C. Udriște, *The Geometry of Metrical Multi-Time Lagrange Spaces*, <http://xxx.lanl.gov/math.DG/0009071>, 2000.
- [12] M. Neagu, C. Udriște, *Torsions and Curvatures on Jet Fibre Bundle $J^1(T, M)$* , <http://xxx.lanl.gov/math.DG/0009069>, 2000.
- [13] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1986.
- [14] D. Saunders, *The Geometry of Jet Bundle*, Cambridge University Press, New York, London, 1989.
- [15] C. Udriște, *Solutions of DEs and PDEs as potential maps using first order Lagrangians*, Centenial Vranceanu, Romanian Academy, University of Bucharest, June 30-July 4, 2000; <http://xxx.lanl.gov/math.DS/0007061>, 2000.
- [16] C. Udriște, M. Neagu, *Geometrical Interpretation of Solutions of Certain PDEs*, Balkan Journal of Geometry and Its Applications, 4,1 (1999), 145-152.

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